

# KAZHDAN'S PROPERTY $T$ FOR DISCRETE QUANTUM GROUPS

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## Abstract

We give a simple definition of property  $T$  for discrete quantum groups. We prove the basic expected properties: discrete quantum groups with property  $T$  are finitely generated and unimodular. Moreover we show that, for "I.C.C." discrete quantum groups, property  $T$  is equivalent to Connes' property  $T$  for the dual von Neumann algebra. This allows us to give the first example of a property  $T$  discrete quantum group which is not a group using the twisting construction.

## 1 Introduction

In the 1980's, Woronowicz [19], [20], [21] introduced the notion of a compact quantum group and generalized the classical Peter-Weyl representation theory. Many interesting examples of compact quantum groups are available by now: Drinfel'd and Jimbo [5], [9] introduced  $q$ -deformations of compact semi-simple Lie groups, and Rosso [13] showed that they fit into the theory of Woronowicz. Free orthogonal and unitary quantum groups were introduced by Van Daele and Wang [18] and studied in detail by Banica [1], [2].

Some discrete group-like properties and proofs have been generalized to (the dual of) compact quantum groups. See, for example, the work of Tomatsu [14] on amenability, the work of Banica and Vergnioux [3] on growth and the work of Vergnioux and Vaes [15] on boundary.

The aim of this paper is to define property  $T$  for discrete quantum groups. We give a definition analogous to the group case using almost invariant vectors. We show that a discrete quantum group with property  $T$  is finitely generated, i.e. the dual is a compact quantum group of matrices. Recall that a locally compact group with property  $T$  is unimodular. We show that the same result holds for discrete quantum groups, i.e. every discrete quantum group with property  $T$  is a Kac algebra. In [4] Connes and Jones defined property  $T$  for arbitrary von Neumann algebras and showed that an I.C.C. group has property  $T$  if and only if its group von Neumann algebra (which is a  $\text{II}_1$  factor) has property  $T$ . We show that if the group von Neumann algebra of a discrete quantum group  $\widehat{\mathbb{G}}$  is

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an infinite dimensional factor (i.e.  $\widehat{\mathbb{G}}$  is “I.C.C.”), then  $\widehat{\mathbb{G}}$  has property  $T$  if and only if its group von Neumann algebra is a  $\text{II}_1$  factor with property  $T$ . This allows us to construct an example of a discrete quantum group with property  $T$  which is not a group by twisting an I.C.C. property  $T$  group. In addition we show that free quantum groups do not have property  $T$ .

This paper is organized as follows: in Section 2 we recall the notions of compact and discrete quantum groups and the main results of this theory. We introduce the notion of discrete quantum sub-groups and prove some basic properties of the quasi-regular representation. We also recall the definition of property  $T$  for von Neumann algebras. In Section 3 we introduce property  $T$  for discrete quantum groups, we give some basic properties and we show our main result.

## 2 Preliminaries

### 2.1 Notations

The scalar product of a Hilbert space  $H$ , which is denoted by  $\langle \cdot, \cdot \rangle$ , is supposed to be linear in the first variable. The von Neumann algebra of bounded operators on  $H$  will be denoted by  $\mathcal{B}(H)$  and the  $C^*$  algebra of compact operators by  $\mathcal{K}_0(H)$ . We will use the same symbol  $\otimes$  to denote the tensor product of Hilbert spaces, the minimal tensor product of  $C^*$  algebras and the spatial tensor product of von Neumann algebras. We will use freely the leg numbering notation.

### 2.2 Compact quantum groups

We briefly overview the theory of compact quantum groups developed by Woronowicz in [21]. We refer to the survey paper [12] for a smooth approach to these results.

**Definition 1.** A compact quantum group is a pair  $\mathbb{G} = (A, \Delta)$ , where  $A$  is a unital  $C^*$  algebra;  $\Delta$  is unital  $*$ -homomorphism from  $A$  to  $A \otimes A$  satisfying  $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$  and  $\Delta(A)(A \otimes 1)$  and  $\Delta(A)(1 \otimes A)$  are dense in  $A \otimes A$ .

**Notation 1.** We denote by  $C(\mathbb{G})$  the  $C^*$  algebra  $A$ .

The major results in the general theory of compact quantum groups are the existence and uniqueness of the Haar state and the Peter-Weyl representation theory.

**Theorem 1.** Let  $\mathbb{G}$  be a compact quantum group. There exists a unique state  $\varphi$  on  $C(\mathbb{G})$  such that  $(\text{id} \otimes \varphi)\Delta(a) = \varphi(a)1 = (\varphi \otimes \text{id})\Delta(a)$  for all  $a \in C(\mathbb{G})$ . The state  $\varphi$  is called the Haar state of  $\mathbb{G}$ .

**Notation 2.** The Haar state need not be faithful. We denote by  $\mathbb{G}_{\text{red}}$  the reduced quantum group obtained by taking  $C(\mathbb{G}_{\text{red}}) = C(\mathbb{G})/I$  where  $I = \{x \in A \mid \varphi(x^*x) = 0\}$ . The Haar measure is faithful on  $\mathbb{G}_{\text{red}}$ . We denote by  $L^\infty(\mathbb{G})$  the von Neumann algebra generated by the G.N.S. representation of the Haar state of  $\mathbb{G}$ . Note that  $L^\infty(\mathbb{G}_{\text{red}}) = L^\infty(\mathbb{G})$ .

**Definition 2.** A unitary representation  $u$  of a compact quantum group  $\mathbb{G}$  on a Hilbert space  $H$  is a unitary element  $u \in M(\mathcal{B}_0(H) \otimes C(\mathbb{G}))$  satisfying

$$(\text{id} \otimes \Delta)(u) = u_{12}u_{13}.$$

Let  $u^1$  and  $u^2$  be two unitary representations of  $\mathbb{G}$  on the respective Hilbert spaces  $H_1$  and  $H_2$ . We define the set of *intertwiners*

$$\text{Mor}(u^1, u^2) = \{T \in \mathcal{B}(H_1, H_2) \mid (T \otimes 1)u^1 = u^2(T \otimes 1)\}.$$

A unitary representation  $u$  is said to be *irreducible* if  $\text{Mor}(u, u) = \mathbb{C}1$ . Two unitary representations  $u^1$  and  $u^2$  are said to be *unitarily equivalent* if there is a unitary element in  $\text{Mor}(u^1, u^2)$ .

**Theorem 2.** Every irreducible representation is finite-dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducibles.

**Definition 3.** Let  $u^1$  and  $u^2$  be unitary representations of  $\mathbb{G}$  on the respective Hilbert spaces  $H_1$  and  $H_2$ . We define the tensor product

$$u^1 \otimes u^2 = u_{13}^1 u_{23}^2 \in M(\mathcal{B}_0(H_1 \otimes H_2) \otimes C(\mathbb{G})).$$

**Notation 3.** We denote by  $\text{Irred}(\mathbb{G})$  the set of (equivalence classes) of irreducible unitary representations of a compact quantum group  $\mathbb{G}$ . For every  $x \in \text{Irred}(\mathbb{G})$  we choose representatives  $u^x$  on the Hilbert space  $H_x$ . Whenever  $x, y \in \text{Irred}(\mathbb{G})$ , we use  $x \otimes y$  to denote the (class of the) unitary representation  $u^x \otimes u^y$ . The class of the trivial representation is denoted by 1.

The set  $\text{Irred}(\mathbb{G})$  is equipped with a natural involution  $x \mapsto \bar{x}$  such that  $u^{\bar{x}}$  is the unique (up to unitary equivalence) irreducible representation such that

$$\text{Mor}(1, x \otimes \bar{x}) \neq 0 \neq \text{Mor}(1, \bar{x} \otimes x).$$

This means that  $x \otimes \bar{x}$  and  $\bar{x} \otimes x$  contain a non-zero invariant vector. Let  $E_x \in H_x \otimes H_{\bar{x}}$  be a non-zero invariant vector and  $J_x$  the invertible antilinear map from  $H_x$  to  $H_{\bar{x}}$  defined by

$$\langle J_x \xi, \eta \rangle = \langle E_x, \xi \otimes \eta \rangle, \quad \text{for all } \xi \in H_x, \eta \in H_{\bar{x}}.$$

Let  $Q_x = J_x^* J_x$ . We will always choose  $E_x$  and  $E_{\bar{x}}$  normalized such that  $\|E_x\| = \|E_{\bar{x}}\|$  and  $J_{\bar{x}} = J_x^{-1}$ . Then  $Q_x$  is uniquely determined,  $\text{Tr}(Q_x) = \|E_x\|^2 = \text{Tr}(Q_x^{-1})$  and  $Q_{\bar{x}} = (J_x J_x^*)^{-1}$ .  $\text{Tr}(Q_x)$  is called the *quantum dimension* of  $x$  and is denoted by  $\dim_q(x)$ . The unitary representation  $u^{\bar{x}}$  is called the *contragredient* of  $u^x$ .

The G.N.S. representation of the Haar state is given by  $(L^2(\mathbb{G}), \Omega)$  where  $L^2(\mathbb{G}) = \bigoplus_{x \in \text{Irred}(\mathbb{G})} H_x \otimes H_{\bar{x}}$ ,  $\Omega \in H_1 \otimes H_{\bar{1}}$  is the unique norm one vector, and

$$(\omega_{\xi, \eta} \otimes \text{id})(u^x) \Omega = \frac{1}{\|E_x\|} \xi \otimes J_x(\eta), \quad \text{for all } \xi, \eta \in H_x.$$

It is easy to see that  $\varphi$  is a trace if and only if  $Q_x = \text{id}$  for all  $x \in \text{Irred}(\mathbb{G})$ . In this case  $\|E_x\| = \sqrt{n_x}$  where  $n_x$  is the dimension of  $H_x$  and  $J_x$  is an anti-unitary operator.

**Notation 4.** Let  $C(\mathbb{G})_s$  be the vector space spanned by the coefficients of all irreducible representations of  $\mathbb{G}$ . Then  $C(\mathbb{G})_s$  is a dense unital  $*$ -subalgebra of  $C(\mathbb{G})$ . Let  $C(\mathbb{G}_{\max})$  be the maximal  $C^*$  completion of the unital  $*$ -algebra  $C(\mathbb{G})_s$ .  $C(\mathbb{G}_{\max})$  has a canonical structure of a compact quantum group. This quantum group is denoted by  $\mathbb{G}_{\max}$  and it is called the *maximal quantum group*.

A morphism of compact quantum groups  $\pi : \mathbb{G} \rightarrow \mathbb{H}$  is a unital  $*$ -homomorphism from  $C(\mathbb{G}_{\max})$  to  $C(\mathbb{H}_{\max})$  such that  $\Delta_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}$ , where  $\Delta_{\mathbb{G}}$  and  $\Delta_{\mathbb{H}}$  denote the comultiplications for  $\mathbb{G}_{\max}$  and  $\mathbb{H}_{\max}$  respectively. We will need the following easy Lemma.

**Lemma 1.** *Let  $\pi$  be a surjective morphism of compact quantum group from  $\mathbb{G}$  to  $\mathbb{H}$  and  $\tilde{\pi}$  be the surjective  $*$ -homomorphism from  $C(\mathbb{G}_{\max})$  to  $C(\mathbb{H})$  obtained by composition of  $\pi$  with the canonical surjection  $C(\mathbb{H}_{\max}) \rightarrow C(\mathbb{H})$ . Then for every irreducible unitary representation  $v$  of  $\mathbb{H}$  there exists an irreducible unitary representation  $u$  of  $\mathbb{G}$  such that  $v$  is contained in the unitary representation  $(\text{id} \otimes \tilde{\pi})(u)$ .*

*Proof.* Let  $\varphi$  be the Haar state of  $\mathbb{H}$  and  $v$  be an irreducible unitary representation of  $\mathbb{H}$  on the Hilbert space  $H_v$ . Because  $v$  is irreducible it is sufficient to show that there exists a unitary irreducible representation  $u$  of  $\mathbb{G}$  such that  $\text{Mor}(w, v) \neq \{0\}$ , where  $w = (\text{id} \otimes \tilde{\pi})(u)$ . Suppose that the statement is false. Then for all irreducible unitary representations  $u$  of  $\mathbb{G}$  on  $H_u$ , we have  $\text{Mor}(w, v) = \{0\}$ . By [12], Lemma 6.3, for every operator  $a : H_v \rightarrow H_u$  the operator  $(\text{id} \otimes \varphi)(v^*(a \otimes 1)w)$  is in  $\text{Mor}(w, v)$ . It follows that for every irreducible unitary representation  $u$  of  $\mathbb{G}$  and every operator  $a : H_v \rightarrow H_u$  we have  $(\text{id} \otimes \varphi)(v^*(a \otimes 1)w) = 0$ . Using the same techniques as in [12], Theorem 6.7, (because, by the surjectivity of  $\pi$ ,  $\tilde{\pi}(C(\mathbb{G})_s)$  is dense in  $C(\mathbb{H})$ ) we find  $(\text{id} \otimes \varphi)(v^*v) = 0$ . But this is a contradiction as  $v^*v = 1$ .  $\square$

The collection of all finite-dimensional unitary representations (given with the concrete Hilbert spaces) of a compact quantum group  $\mathbb{G}$  is a *complete concrete monoidal  $W^*$ -category*. We denote this category by  $\mathcal{R}(\mathbb{G})$ . We say that  $\mathcal{R}(\mathbb{G})$  is finitely generated if there exists a finite subset  $E \subset \text{Irred}(\mathbb{G})$  such that for all finite-dimensional unitary representations  $r$  there exists a finite family of morphisms  $b_k \in \text{Mor}(r_k, r)$ , where  $r_k$  is a product of elements of  $E$ , and  $\sum_k b_k b_k^* = I_r$ . It is not difficult to show that  $\mathcal{R}(\mathbb{G})$  is finitely generated if and only if  $\mathbb{G}$  is a compact quantum group of matrices (see [20]).

## 2.3 Discrete quantum groups

A discrete quantum group is defined as the dual of a compact quantum group.

**Definition 4.** Let  $\mathbb{G}$  be a compact quantum group. We define the dual *discrete quantum group*  $\widehat{\mathbb{G}}$  as follows:

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{x \in \text{Irred}(\mathbb{G})}^{c_0} \mathcal{B}(H_x), \quad l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{x \in \text{Irred}(\mathbb{G})}^{\infty} \mathcal{B}(H_x).$$

We denote the minimal central projection of  $l^\infty(\widehat{\mathbb{G}})$  by  $p_x$ ,  $x \in \text{Irred}(\mathbb{G})$ . We have a natural unitary  $\mathbb{V} \in M(c_o(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$  given by

$$\mathbb{V} = \bigoplus_{x \in \text{Irred}(\mathbb{G})} u^x.$$

We have a natural comultiplication

$$\hat{\Delta} : l^\infty(\widehat{\mathbb{G}}) \rightarrow l^\infty(\widehat{\mathbb{G}}) \otimes l^\infty(\widehat{\mathbb{G}}) : (\hat{\Delta} \otimes \text{id})(\mathbb{V}) = \mathbb{V}_{13} \mathbb{V}_{23}.$$

The comultiplication is given by the following formula

$$\hat{\Delta}(a)S = Sa, \quad \text{for all } a \in \mathcal{B}(H_x), S \in \text{Mor}(x, yz), x, y, z \in \text{Irred}(\mathbb{G}).$$

**Remark 1.** The maximal and reduced versions of a compact quantum group are different versions of the same underlying compact quantum group. This different versions give the same dual discrete quantum group, i.e.  $\widehat{\mathbb{G}} = \widehat{\mathbb{G}_{\text{red}}} = \widehat{\mathbb{G}_{\text{max}}}$ . This means that  $\widehat{\mathbb{G}}$ ,  $\widehat{\mathbb{G}_{\text{red}}}$  and  $\widehat{\mathbb{G}_{\text{max}}}$  have the same  $C^*$  algebra, the same von Neumann algebra and the same comultiplication.

A morphism of discrete quantum groups  $\hat{\pi} : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{H}}$  is a non-degenerate  $*$ -homomorphism from  $c_0(\widehat{\mathbb{G}})$  to  $M(c_0(\widehat{\mathbb{H}}))$  such that  $\hat{\Delta}_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \hat{\Delta}_{\mathbb{G}}$ , where  $\hat{\Delta}_{\mathbb{G}}$  and  $\hat{\Delta}_{\mathbb{H}}$  denote the comultiplication for  $\widehat{\mathbb{G}}$  and  $\widehat{\mathbb{H}}$  respectively. Every morphism of compact quantum groups  $\pi : \mathbb{G} \rightarrow \mathbb{H}$  admits a canonical dual morphism of discrete quantum groups  $\hat{\pi} : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{H}}$ . Conversely, every morphism of discrete quantum groups  $\hat{\pi} : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{H}}$  admits a canonical dual morphism of compact quantum groups  $\pi : \mathbb{G} \rightarrow \mathbb{H}$ . Moreover,  $\pi$  is surjective (resp. injective) if and only if  $\hat{\pi}$  is injective (resp. surjective).

We say that a discrete quantum group  $\widehat{\mathbb{G}}$  is *finitely generated* if the category  $\mathcal{R}(\widehat{\mathbb{G}})$  is finitely generated.

We will work with representations in the von Neumann algebra setting.

**Definition 5.** Let  $\widehat{\mathbb{G}}$  be a discrete quantum group. A *unitary representation*  $U$  of  $\widehat{\mathbb{G}}$  on a Hilbert space  $H$  is a unitary  $U \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(H)$  such that :

$$(\hat{\Delta} \otimes \text{id})(U) = U_{13} U_{23}.$$

Consider the following maximal version of the unitary  $\mathbb{V}$ :

$$\mathcal{V} = \bigoplus_{x \in \text{Irred}(\mathbb{G})} u^x \in M(c_o(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}_{\text{max}})).$$

For every unitary representation  $U$  of  $\widehat{\mathbb{G}}$  on a Hilbert space  $H$  there exists a unique  $*$ -homomorphism  $\rho : C(\mathbb{G}_{\text{max}}) \rightarrow \mathcal{B}(H)$  such that  $(\text{id} \otimes \rho)(\mathcal{V}) = U$ .

**Notation 5.** Whenever  $U$  is a unitary representation of  $\widehat{\mathbb{G}}$  on a Hilbert space  $H$  we write  $U = \sum_{x \in \text{Irred}(\mathbb{G})} U^x$  where  $U^x = U p_x$  is a unitary in  $\mathcal{B}(H_x) \otimes \mathcal{B}(H)$ .

The discrete quantum group  $l^\infty(\widehat{\mathbb{G}})$  comes equipped with a natural modular structure. Let us define the following canonical states on  $\mathcal{B}(H_x)$ :

$$\varphi_x(A) = \frac{\text{Tr}(Q_x A)}{\text{Tr}(Q_x)}, \quad \text{and} \quad \psi_x(A) = \frac{\text{Tr}(Q_x^{-1} A)}{\text{Tr}(Q_x^{-1})}, \quad \text{for all } A \in \mathcal{B}(H_x).$$

The states  $\varphi_x$  and  $\psi_x$  provide a formula for the invariant normal semi-finite faithful (n.s.f.) weights on  $l^\infty(\widehat{\mathbb{G}})$ .

**Proposition 1.** *The left invariant weight  $\hat{\varphi}$  and the right invariant weight  $\hat{\psi}$  on  $\widehat{\mathbb{G}}$  are given by*

$$\hat{\varphi}(a) = \sum_{x \in \text{Irred}(\mathbb{G})} \dim_q(x)^2 \varphi_x(ap_x) \quad \text{and} \quad \hat{\psi}(a) = \sum_{x \in \text{Irred}(\mathbb{G})} \dim_q(x)^2 \psi_x(ap_x),$$

for all  $a \in l^\infty(\widehat{\mathbb{G}})$  whenever this formula makes sense.

A discrete quantum is unimodular (i.e. the left and right invariant weights are equal) if and only if the Haar state  $\varphi$  on the dual is a trace. In general, a discrete quantum group is not unimodular, and it is easy to check that the Radon-Nikodym derivative is given by

$$[D\hat{\psi} : D\hat{\varphi}]_t = \hat{\delta}^{it} \quad \text{where} \quad \hat{\delta} = \sum_{x \in \text{Irred}(\mathbb{G})} Q_x^{-2} p_x.$$

The positive self-adjoint operator  $\hat{\delta}$  is called the *modular element*: it is affiliated with  $c_0(\widehat{\mathbb{G}})$  and satisfies  $\hat{\Delta}(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta}$ .

The following Proposition is very easy to prove.

**Proposition 2.** *Let  $\Gamma$  be the subset of  $\mathbb{R}_+^*$  consisting of all the eigenvalues of the operators  $Q_x^{-2}$  for  $x \in \text{Irred}(\mathbb{G})$ . Then  $\Gamma$  is a subgroup of  $\mathbb{R}_+^*$  and  $Sp(\hat{\delta}) = \Gamma \cup \{0\}$ .*

*Proof.* Note that, because  $J_{\bar{x}} = J_x^{-1}$ , the eigenvalues of  $Q_{\bar{x}}$  are the inverse of the eigenvalues of  $Q_x$ . Using the formula  $SQ_z = Q_x \otimes Q_y S$ , when  $z \subset x \otimes y$  and  $S \in \text{Mor}(z, x \otimes y)$  is an isometry, the Proposition follows immediately.  $\square$

## 2.4 Discrete quantum subgroups

Let  $\mathbb{G}$  be a compact quantum group with representation category  $\mathcal{C}$ . Let  $\mathcal{D}$  be a full subcategory such that  $1_{\mathcal{C}} \in \mathcal{D}$ ,  $\mathcal{D} \otimes \mathcal{D} \subset \mathcal{D}$  and  $\overline{\mathcal{D}} = \mathcal{D}$ . By the Tannaka-Krein Reconstruction Theorem of Woronowicz [20] we know that there exists a compact quantum group  $\mathbb{H}$  such that the representation category of  $\mathbb{H}$  is  $\mathcal{D}$ . We say that  $\mathbb{H}$  is a *discrete quantum subgroup* of  $\widehat{\mathbb{G}}$ . We have  $\text{Irred}(\mathbb{H}) \subset \text{Irred}(\mathbb{G})$ . We collect some easy observations in the next proposition. We denote by a subscript  $\mathbb{H}$  the objects associated to  $\mathbb{H}$ .

**Proposition 3.** *Let  $p = \sum_{x \in \text{Irred}(\mathbb{H})} p_x$ . We have:*

1.  $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$ ;
2.  $l^\infty(\hat{\mathbb{H}}) = p(l^\infty(\hat{\mathbb{G}}))$ ;
3.  $\hat{\Delta}_{\mathbb{H}}(a) = \hat{\Delta}(a)(p \otimes p)$  for all  $a \in l^\infty(\hat{\mathbb{H}})$ ;
4.  $\hat{\varphi}(p) = \hat{\varphi}_{\mathbb{H}}$  and  $\hat{\delta}_{\mathbb{H}} = p\hat{\delta}$ .

*Proof.* For  $x, y, z \in \text{Irred}(\mathbb{G})$  such that  $y \subset z \otimes x$ , we denote by  $p_y^{z \otimes x} \in \text{End}(x \otimes y)$  the projection on the sum of all sub-representations equivalent to  $y$ . Note that

$$\hat{\Delta}(p_y)(p_z \otimes p_x) = \begin{cases} p_y^{z \otimes x} & \text{if } y \subset z \otimes x, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Thus:

$$\hat{\Delta}(p)(p_z \otimes p_x) = \sum_{y \in \text{Irred}(\mathbb{H}), y \subset z \otimes x} p_y^{z \otimes x}.$$

Note that if  $y \subset z \otimes x$  and  $y, z \in \text{Irred}(\mathbb{H})$  then  $x \in \text{Irred}(\mathbb{H})$ . It follows that:

$$\hat{\Delta}(p)(p \otimes p_x) = \begin{cases} p \otimes p_x & \text{if } x \in \text{Irred}(\mathbb{H}), \\ 0 & \text{otherwise.} \end{cases}$$

Thus,  $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$ . The other assertions are obvious.  $\square$

We introduce the following equivalence relation on  $\text{Irred}(\mathbb{G})$  (see [17]): if  $x, y \in \text{Irred}(\mathbb{G})$  then  $x \sim y$  if and only if there exists  $t \in \text{Irred}(\mathbb{H})$  such that  $x \subset y \otimes t$ . We define the right action of  $\hat{\mathbb{H}}$  on  $l^\infty(\hat{\mathbb{G}})$  by translation:

$$\alpha : l^\infty(\hat{\mathbb{G}}) \rightarrow l^\infty(\hat{\mathbb{G}}) \otimes l^\infty(\hat{\mathbb{H}}), \quad \alpha(a) = \hat{\Delta}(a)(1 \otimes p).$$

Using  $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$  and  $\hat{\Delta}_{\mathbb{H}} = \hat{\Delta}(\cdot)(p \otimes p)$  it is easy to see that  $\alpha$  satisfies the following equations:

$$(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \hat{\Delta}_{\mathbb{H}})\alpha \quad \text{and} \quad (\hat{\Delta} \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\hat{\Delta}.$$

The first equality means that  $\alpha$  is a right action of  $\hat{\mathbb{H}}$  on  $l^\infty(\hat{\mathbb{G}})$ . Let  $l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}})$  be the set of fixed points of the action  $\alpha$ :

$$l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}}) := \{a \in l^\infty(\hat{\mathbb{G}}), \alpha(a) = a \otimes 1\}.$$

Using the second equality for  $\alpha$  it is easy to see that:

$$\hat{\Delta}(l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}})) \subset l^\infty(\hat{\mathbb{G}}) \otimes l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}}).$$

Thus the restriction of  $\hat{\Delta}$  to  $l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}})$  gives an action of  $\hat{\mathbb{G}}$  on  $l^\infty(\hat{\mathbb{G}}/\hat{\mathbb{H}})$ . We denote this action by  $\beta$ .

**Proposition 4.** Let  $T_\alpha = (\text{id} \otimes \hat{\varphi}_\mathbb{H})\alpha$  be the normal faithful operator valued weight from  $l^\infty(\widehat{\mathbb{G}})$  to  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  associated to  $\alpha$ .  $T_\alpha$  is semi-finite and there exists a unique n.s.f. weight  $\theta$  on  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  such that  $\hat{\varphi} = \theta \circ T_\alpha$ .

*Proof.* It follows from Eq. (1) that  $T_\alpha(p_y)p_z = 0$  if  $z \not\sim y$ . Take  $z \sim y$ , we have:

$$\begin{aligned} T_\alpha(p_y)p_z &= \sum_{x \in \text{Irred}(\mathbb{H})} \dim_q(x)^2 (\text{id} \otimes \varphi_x)(p_y^{z \otimes x}) \\ &\leq \sum_{x \in \text{Irred}(\mathbb{G})} \dim_q(x)^2 (\text{id} \otimes \varphi_x)(p_y^{z \otimes x}) \\ &= (\text{id} \otimes \hat{\varphi})(\hat{\Delta}(p_y))p_z = \hat{\varphi}(p_y)p_z \\ &= \dim_q(y)^2 p_z. \end{aligned}$$

It follows that  $T_\alpha(p_y) < \infty$  for all  $y$ . This implies that  $T_\alpha$  is semi-finite. Note that  $\alpha(\delta^{-it}) = \delta^{-it} \otimes \delta_\mathbb{H}^{-it}$ . It follows from [10], Proposition 8.7, that there exists a unique n.s.f. weight  $\theta$  on  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  such that  $\hat{\varphi} = \theta \circ T_\alpha$ .  $\square$

Denote by  $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  the G.N.S. space of  $\theta$  and suppose that  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) \subset \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$ . Let  $U^* \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}))$  be the unitary implementation of  $\beta$  associated to  $\theta$  in the sense of [16]. Then  $U$  is a unitary representation of  $\widehat{\mathbb{G}}$  on  $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  and  $\beta(x) = U^*(1 \otimes x)U$ . We call  $U$  the *quasi-regular* representation of  $\widehat{\mathbb{G}}$  modulo  $\widehat{\mathbb{H}}$ .

**Lemma 2.** We have  $p \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}) \cap \mathcal{N}_\theta$ . Put  $\xi = \Lambda_\theta(p)$ . If  $\widehat{\mathbb{G}}$  is unimodular then  $U^x \eta \otimes \xi = \eta \otimes \xi$  for all  $x \in \text{Irred}(\mathbb{H})$  and all  $\eta \in H_x$ .

*Proof.* Using  $\hat{\Delta}(p_1)(1 \otimes p_x) = p_1^{\bar{x} \otimes x}$  it is easy to see that  $T_\alpha(p_1) = p$ . It follows that  $p \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  and  $\theta(p) = \hat{\varphi}(p_1) = 1$ . Thus  $p \in \mathcal{N}_\theta$ . Let  $x \in M^+$  such that  $T_\alpha(x) < \infty$ ,  $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$  and  $\mu$  a n.s.f. weight on  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ . Using  $(\hat{\Delta} \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\hat{\Delta}$  we find:

$$\begin{aligned} (\omega \otimes \mu)\beta(T_\alpha(x)) &= (\omega \otimes \mu)\hat{\Delta}(T_\alpha(x)) = (\omega \otimes \mu)\hat{\Delta}((\text{id} \otimes \hat{\varphi}_\mathbb{H})(\alpha(x))) \\ &= (\omega \otimes \mu \otimes \hat{\varphi}_\mathbb{H})((\hat{\Delta} \otimes \text{id})\alpha(x)) \\ &= (\omega \otimes \mu \otimes \hat{\varphi}_\mathbb{H})((\text{id} \otimes \alpha)\hat{\Delta}(x)) \\ &= (\omega \otimes \mu \circ T_\alpha)\hat{\Delta}(x). \end{aligned} \tag{2}$$

It follows that, for all  $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$  and all  $y \in l^\infty(\widehat{\mathbb{G}})^+$  such that  $T_\alpha(y) < \infty$ , we have:

$$(\omega \otimes \theta)\beta(T_\alpha(y)) = (\omega \otimes \hat{\varphi})(\hat{\Delta}(y)) = \hat{\varphi}(y)\omega(1) = \theta(T_\alpha(y))\omega(1).$$

Let  $x \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})^+$ . Because  $T_\alpha$  is a faithful and semi-finite, there exists an increasing net of positive elements  $y_i$  in  $l^\infty(\widehat{\mathbb{G}})^+$  such that  $T_\alpha(y_i) < \infty$  for all  $i$  and  $\text{Sup}_i(T_\alpha(y_i)) = x$ . It follows that:

$$(\omega \otimes \theta)\beta(x) = \text{Sup}((\omega \otimes \theta)\beta(T_\alpha(y_i))) = \text{Sup}(\theta(T_\alpha(y_i))\omega(1)) = \theta(x)\omega(1),$$



for all  $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$ . This means that  $\theta$  is  $\beta$ -invariant. Using this invariance we define the following isometry:

$$V^*(\hat{\Lambda}(x) \otimes \Lambda_\theta(y)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\beta(y)(x \otimes 1)).$$

Because  $\widehat{\mathbb{G}}$  is unimodular we know from [16], Proposition 4.3, that  $V^*$  is the unitary implementation of  $\beta$  associated to  $\theta$  i.e.  $V = U$ . Using  $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$ , it follows that, for all  $x \in \mathcal{N}_{\hat{\varphi}}$ , we have:

$$U^*(p\hat{\Lambda}(x) \otimes \Lambda_\theta(p)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\hat{\Delta}(p)(px \otimes 1)) = p\hat{\Lambda}(x) \otimes \Lambda_\theta(p).$$

This concludes the proof.  $\square$

**Remark 2.** For general discrete quantum groups it can be proved, as in [6], Théorème 2.9, that  $V^*$  is a unitary implementing the action  $\beta$  and, as in [16], Proposition 4.3, that  $V^*$  is the unitary implementation of  $\beta$  associated to  $\theta$ . Thus the previous lemma is also true for general discrete quantum groups.

**Lemma 3.** *Suppose that  $U$  has a non-zero invariant vector  $\xi \in l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ . Then  $\text{Irred}(\mathbb{G})/\text{Irred}(\mathbb{H})$  is a finite set.*

*Proof.* Let  $\xi \in l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$  be a normalized  $U$ -invariant vector. Using  $\beta(x) = U^*(1 \otimes x)U$  it is easy to see that  $\omega_\xi$  is a  $\beta$ -invariant normal state on  $l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ , i.e.  $(\text{id} \otimes \omega_\xi)\beta(x) = \omega_\xi(x)1$  for all  $x \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}})$ . Let  $s$  be the support of  $\omega_\xi$  and  $e = 1 - s$ . Let  $\omega$  be a faithful normal state on  $l^\infty(\widehat{\mathbb{G}})$ . Because the support of  $\omega \otimes \omega_\xi$  is  $1 \otimes s$  and  $(\omega \otimes \omega_\xi)\beta(e) = \omega_\xi(e) = 0$  we find  $\hat{\Delta}(e) = \beta(e) \leq 1 \otimes e$ . It follows from [11], Lemma 6.4, that  $e = 0$  or  $e = 1$ . Because  $\xi$  is a non-zero vector we have  $e = 0$ . Thus  $\omega_\xi$  is faithful. Let  $x \in M^+$  such that  $T_\alpha(x) < \infty$ . By Eq. (2) we have:

$$(\omega \otimes \omega_\xi \circ T_\alpha)(\hat{\Delta}(x)) = (\omega \otimes \omega_\xi)\beta(T_\alpha(x)) = \omega_\xi(T_\alpha(x))\omega(1),$$

for all  $\omega \in l^\infty(\widehat{\mathbb{G}})_*^+$ . Because  $T_\alpha$  is n.s.f., it follows easily that  $\omega_\xi \circ T_\alpha$  is a left invariant n.s.f. weight on  $\widehat{\mathbb{G}}$ . Thus, up to a positive constant, we have  $\omega_\xi \circ T_\alpha = \hat{\varphi}$ .

Suppose that  $\text{Irred}(\mathbb{G})/\text{Irred}(\mathbb{H})$  is infinite, and let  $x_i \in \text{Irred}(\mathbb{G})$ ,  $i \in \mathbb{N}$  be a complete set of representatives of  $\text{Irred}(\mathbb{G})/\text{Irred}(\mathbb{H})$ . Let  $a$  be the positive element of  $l^\infty(\widehat{\mathbb{G}})$  defined by  $a = \sum_{i \geq 0} \frac{1}{\dim_q(x_i)^2} p_{x_i}$ . Then we have  $\hat{\varphi}(a) = +\infty$  and  $T_\alpha(a) = \sum_i \sum_{x \simeq x_i} p_x = 1 < \infty$ , which is a contradiction.  $\square$

## 2.5 Property $T$ for von Neumann algebras

Here we recall several facts from [4]. If  $M$  and  $N$  are von Neumann algebras then a correspondence from  $M$  to  $N$  is a Hilbert space  $H$  which is both a left  $M$ -module and a right  $N$ -module, with commuting normal actions  $\pi_l$  and  $\pi_r$  respectively. The triple  $(H, \pi_l, \pi_r)$  is simply denoted by  $H$  and we shall write  $a\xi b$  instead of  $\pi_l(a)\pi_r(b)\xi$  for  $a \in M$ ,  $b \in N$  and  $\xi \in H$ . We shall denote by

$\mathcal{C}(M)$  the set of unitary equivalence classes of correspondences from  $M$  to  $M$ . The standard representation of  $M$  defines an element  $L^2(M)$  of  $\mathcal{C}(M)$ , called the identity correspondence.

Given  $H \in \mathcal{C}(M)$ ,  $\epsilon > 0$ ,  $\xi_1, \dots, \xi_n \in H$ ,  $a_1, \dots, a_p \in M$ , let  $\mathcal{V}_H(\epsilon, \xi_i, a_i)$  be the set of  $K \in \mathcal{C}(M)$  for which there exist  $\eta_1, \dots, \eta_n \in K$  with

$$|\langle a_j \eta_i a_k, \eta_{i'} \rangle - \langle a_j \xi_i a_k, \xi_{i'} \rangle| < \epsilon, \quad \text{for all } i, i', j, k.$$

Such sets form a basis of a topology on  $\mathcal{C}(M)$  and, following [4],  $M$  is said to have property  $T$  if there is a neighbourhood of the identity correspondence, each member of which contains  $L^2(M)$  as a direct summand.

When  $M$  is a  $\text{II}_1$  factor the property  $T$  is easier to understand. A  $\text{II}_1$  factor  $M$  has property  $T$  if we can find  $\epsilon > 0$  and  $a_1, \dots, a_p \in M$  satisfying the following condition: every  $H \in \mathcal{C}(M)$  such that there exists  $\xi \in H$ ,  $\|\xi\| = 1$ , with  $\|a_i \xi - \xi a_i\| < \epsilon$  for all  $i$ , contains a non-zero central vector  $\eta$  (i.e.  $a\eta = \eta a$  for all  $a \in M$ ). We recall the following Proposition from [4].

**Proposition 5.** *If  $M$  is a  $\text{II}_1$  factor with property  $T$  then there exist  $\epsilon > 0$ ,  $b_1, \dots, b_m \in M$  and  $C > 0$  with the following property: for any  $\delta \leq \epsilon$ , if  $H \in \mathcal{C}(M)$  and  $\xi \in H$  is a unit vector satisfying  $\|b_i \xi - \xi b_i\| < \delta$  for all  $1 \leq i \leq m$ , then there exists a unit central vector  $\eta \in H$  such that  $\|\xi - \eta\| < C\delta$ .*

It is proved in [4] that a discrete I.C.C. group has property  $T$  if and only if the group von Neumann algebra  $\mathcal{L}(G)$  has property  $T$ .

### 3 Property $T$ for Discrete Quantum Groups

**Definition 6.** Let  $\widehat{\mathbb{G}}$  be a discrete quantum group.

- Let  $E \subset \text{Irred}(\widehat{\mathbb{G}})$  be a finite subset,  $\epsilon > 0$  and  $U$  a unitary representation of  $\widehat{\mathbb{G}}$  on a Hilbert space  $K$ . We say that  $U$  has an  $(E, \epsilon)$ -invariant vector if there exists a unit vector  $\xi \in K$  such that for all  $x \in E$  and  $\eta \in H_x$  we have:

$$\|U^x \eta \otimes \xi - \eta \otimes \xi\| < \epsilon \|\eta\|.$$

- We say that  $U$  has *almost invariant vectors* if, for all finite subsets  $E \subset \text{Irred}(\widehat{\mathbb{G}})$  and all  $\epsilon > 0$ ,  $U$  has an  $(E, \epsilon)$ -invariant vector.
- We say that  $\widehat{\mathbb{G}}$  has *property  $T$*  if every unitary representation of  $\widehat{\mathbb{G}}$  having almost invariant vectors has a non-zero invariant vector.

**Remark 3.** Let  $\mathbb{G} = (C^*(\Gamma), \Delta)$ , where  $\Gamma$  is a discrete group and  $\Delta(g) = g \otimes g$  for  $g \in \Gamma$ . It follows from the definition that  $\widehat{\mathbb{G}}$  has property  $T$  if and only if  $\Gamma$  has property  $T$ .

The next proposition will be useful to show that the dual of a free quantum group does not have property  $T$ .

**Proposition 6.** *Let  $\mathbb{G}$  and  $\mathbb{H}$  be compact quantum groups. Suppose that there is a surjective morphism of compact quantum groups from  $\mathbb{G}$  to  $\mathbb{H}$  (or an injective morphism of discrete quantum groups from  $\widehat{\mathbb{H}}$  to  $\widehat{\mathbb{G}}$ ). If  $\widehat{\mathbb{G}}$  has property  $T$  then  $\widehat{\mathbb{H}}$  has property  $T$ .*

*Proof.* We can suppose that  $\mathbb{G} = \mathbb{G}_{\max}$  and  $\mathbb{H} = \mathbb{H}_{\max}$ . We will denote by a subscript  $\mathbb{G}$  (resp.  $\mathbb{H}$ ) the object associated to  $\mathbb{G}$  (resp.  $\mathbb{H}$ ). Let  $\pi$  be the surjective morphism from  $C(\mathbb{G})$  to  $C(\mathbb{H})$  which intertwines the comultiplications. Let  $U$  be a unitary representation of  $\mathbb{H}$  on a Hilbert space  $K$  and suppose that  $U$  has almost invariant vectors. Let  $\rho$  be the unique morphism from  $C(\mathbb{H})$  to  $\mathcal{B}(K)$  such that  $(\text{id} \otimes \rho)(\mathcal{V}_{\mathbb{H}}) = U$ . Consider the following unitary representation of  $\widehat{\mathbb{G}}$  on  $K$ :  $V = (\text{id} \otimes (\rho \circ \pi))(\mathcal{V}_{\mathbb{G}})$ . We will show that  $V$  has almost invariant vectors. Let  $E \subset \text{Irred}(\mathbb{G})$  be a finite subset and  $\epsilon > 0$ . For  $x \in \text{Irred}(\mathbb{G})$  and  $y \in \text{Irred}(\mathbb{H})$  denote by  $u^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$  and  $v^y \in \mathcal{B}(H_y) \otimes C(\mathbb{H})$  a representative of  $x$  and  $y$  respectively. Note that  $w^x = (\text{id} \otimes \pi)(u^x)$  is a finite dimensional unitary representation of  $\mathbb{H}$ , thus we can suppose that  $w^x = \oplus_{n_{x,y}} v^y$ . Let  $L = \{y \in \text{Irred}(\mathbb{H}) \mid \exists x \in E, n_{x,y} \neq 0\}$ . Because  $U$  has almost invariant vectors, there exists a norm one vector  $\xi \in K$  such that  $\|U^y \eta \otimes \xi - \eta \otimes \xi\| < \epsilon \|\eta\|$  for all  $y \in L$  and all  $\eta \in H_y$ . Using the isomorphism

$$H_x = \bigoplus_{y \in \text{Irred}(\mathbb{H}), n_{x,y} \neq 0} \underbrace{(H_y \oplus \dots \oplus H_y)}_{n_{x,y}},$$

we can identify  $V^x$  with  $\oplus_{n_{x,y}} U^y$  in  $\bigoplus_y \mathcal{B}(H_y) \oplus \mathcal{B}(H_y) \oplus \dots \oplus \mathcal{B}(H_y) \otimes \mathcal{B}(K)$ . With this identification it is easy to see that, for all  $x \in E$  and all  $\eta$  in  $H_x$ , we have  $\|V^x \eta \otimes \xi - \eta \otimes \xi\| < \epsilon \|\eta\|$ . It follows that  $V$  has almost invariant vectors and thus there is a non-zero  $V$ -invariant vector, say  $l$ , in  $K$ . To show that  $l$  is also  $U$ -invariant it is sufficient to show that for every  $y \in \text{Irred}(\mathbb{H})$  there exists  $x \in \text{Irred}(\mathbb{G})$  such that  $n_{x,y} \neq 0$ . This follows from Lemma 1.  $\square$

**Corollary 1.** *The discrete quantum groups  $\widehat{A_o(n)}$ ,  $\widehat{A_u(n)}$  and  $\widehat{A_s(n)}$  do not have property  $T$  for  $n \geq 2$ .*

*Proof.* It follows directly from the preceding proposition and the following surjective morphisms:

$$A_o(n) \rightarrow C^*(\star_{i=1}^n \mathbb{Z}_2), \quad A_u(n) \rightarrow C^*(\mathbb{F}_n), \quad A_s(n) \rightarrow C^*(\star_{i=1}^n \mathbb{Z}_{n_i}),$$

where  $\sum n_i = n$ .  $\square$

In the next Proposition we show that discrete quantum groups with property  $T$  are unimodular.

**Proposition 7.** *Let  $\widehat{\mathbb{G}}$  be a discrete quantum group. If  $\widehat{\mathbb{G}}$  has property  $T$  then it is a Kac algebra, i.e. the Haar state  $\varphi$  on  $\mathbb{G}$  is a trace.*

*Proof.* Suppose  $\widehat{\mathbb{G}}$  has property  $T$  and let  $\Gamma$  be the discrete group introduced in Proposition 2. Because  $\text{Sp}(\widehat{\delta}) = \Gamma \cup \{0\}$  and  $\widehat{\Delta}\widehat{\delta} = \widehat{\delta} \otimes \widehat{\delta}$ , we have an injective  $*$ -homomorphism

$$\alpha : c_0(\Gamma) \rightarrow c_0(\widehat{\mathbb{G}}), \quad \alpha(f) = f(\widehat{\delta})$$

satisfying  $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_\Gamma$ . By Proposition 6,  $\Gamma$  has property  $T$ . It follows that  $\Gamma = \{1\}$  and  $\widehat{\delta} = 1$ . Thus  $Q_x = 1$  for all  $x \in \text{Irred}(\mathbb{G})$ . This means that  $\varphi$  is a trace.  $\square$

**Proposition 8.** *Let  $\widehat{\mathbb{G}}$  be a discrete quantum group. If  $\widehat{\mathbb{G}}$  has property  $T$  then it is finitely generated.*

*Proof.* Let  $\text{Irred}(\mathbb{G}) = \{x_n \mid n \in \mathbb{N}\}$  and  $\mathcal{C}$  be the category of finite dimensional unitary representations of  $\mathbb{G}$ . For  $i \in \mathbb{N}$  let  $\mathcal{D}_i$  be the full subcategory of  $\mathcal{C}$  generated by  $(x_0, \dots, x_i)$ . This means that the irreducibles of  $\mathcal{D}_i$  are the irreducible representations  $u$  of  $\mathbb{G}$  such that  $u$  is equivalent to a sub-representation of  $x_{k_1}^{\epsilon_1} \otimes \dots \otimes x_{k_l}^{\epsilon_l}$  for  $l \geq 1$ ,  $0 \leq k_j \leq i$ , and  $\epsilon_j$  is nothing or the contragredient. The Hilbert spaces and the morphisms are the same in  $\mathcal{D}_i$  or in  $\mathcal{D}$ . Thus we have  $1_{\mathcal{C}} \in \mathcal{D}_i$ ,  $\mathcal{D}_i \otimes \mathcal{D}_i \subset \mathcal{D}_i$  and  $\overline{\mathcal{D}_i} = \mathcal{D}_i$ . Let  $\mathbb{H}_i$  be the compact quantum group such that  $\mathcal{D}_i$  is the category of representation of  $\mathbb{H}_i$ . Let  $U_i \in l^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i) \otimes \mathcal{B}(l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i))$  be the quasi-regular representation of  $\widehat{\mathbb{G}}$  modulo  $\widehat{\mathbb{H}}_i$ . Let  $U$  be the direct sum of the  $U_i$ ; this a unitary representation on  $K = \bigoplus l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_i)$ . Let us show that  $U$  has almost invariant vectors. Let  $E \subset \text{Irred}(\mathbb{G})$  be a finite subset. There exists  $i_0$  such that  $E \subset \text{Irred}(\mathbb{H}_{i_0})$  for all  $i \geq i_0$ . By Lemma 2 we have a unit vector  $\xi$  in  $l^2(\widehat{\mathbb{G}}/\widehat{\mathbb{H}}_{i_0})$  such that  $U_{i_0}^x \eta \otimes \xi = \eta \otimes \xi$  for all  $x \in E$  and all  $\eta \in H_x$ . Let  $\tilde{\xi} = (\xi_i) \in K$  where  $\xi_i = 0$  if  $i \neq i_0$  and  $\xi_{i_0} = \xi$ . Then  $\tilde{\xi}$  is a unit vector in  $K$  such that  $U^x \eta \otimes \tilde{\xi} = \eta \otimes \tilde{\xi}$  for all  $x \in E$ . It follows that  $U$  has an almost invariant vector. By property  $T$  there exists a non-zero invariant vector  $l = (l_i) \in K$ . There exists  $m$  such that  $l_m \neq 0$ . Then  $l_m$  is an invariant vector for  $U_m$ . By Lemma 3,  $\text{Irred}(\mathbb{G})/\text{Irred}(\mathbb{H}_m)$  is a finite set. Let  $y_1, \dots, y_l$  be a complete set of representatives of  $\text{Irred}(\mathbb{G})/\text{Irred}(\mathbb{H}_m)$ . Then  $\mathcal{C}$  is generated by  $\{y_1, \dots, y_l, x_0, \dots, x_m, \bar{x}_0, \dots, \bar{x}_m\}$ .  $\square$

As in the classical case, we can show that property  $T$  is equivalent to the existence of a Kazhdan pair.

**Proposition 9.** *Let  $\widehat{\mathbb{G}}$  be a finitely generated discrete quantum group. Let  $E \subset \text{Irred}(\mathbb{G})$  be a finite subset with  $1 \in E$  such that  $\mathcal{R}(\mathbb{G})$  is generated by  $E$ . The following assertions are equivalent:*

1.  $\widehat{\mathbb{G}}$  has property  $T$ .
2. There exists  $\epsilon > 0$  such that every unitary representation of  $\widehat{\mathbb{G}}$  having an  $(E, \epsilon)$ -invariant vector has a non-zero invariant vector.

*Proof.* It is sufficient to show that 1 implies 2. Let  $n \in \mathbb{N}^*$  and  $E_n = \{y \in \text{Irred}(\mathbb{G}) \mid y \subset x_1 \dots x_n, x_i \in E\}$ . Because  $1 \in E$ , the sequence  $(E_n)_{n \in \mathbb{N}^*}$  is increasing. Let us show that  $\text{Irred}(\mathbb{G}) = \bigcup E_n$ . Let  $r \in \text{Irred}(\mathbb{G})$ . Because  $\mathcal{R}(\mathbb{G})$

is generated by  $E$ , there exists a finite family of morphisms  $b_k \in \text{Mor}(r_k, r)$ , where  $r_k$  is a product of elements of  $E$  and  $\sum_k b_k b_k^* = I_r$ . Let  $L$  be the maximum of the length of the elements  $r_k$ . Because  $1 \in E$ , we can suppose that all the  $r_k$  are of the form  $x_1 \dots x_L$  with  $x_i \in E$ . Put  $t_k = b_k^*$ . Note that  $t_k^* t_k \in \text{Mor}(r, r)$ . Because  $r$  is irreducible and  $\sum_k t_k^* t_k = I_r$ , there exists a unique  $k$  such that  $t_k^* t_k = I_r$  and  $t_l^* t_l = 0$  if  $l \neq k$ . Thus  $t_k \in \text{Mor}(r, r_k)$  is an isometry. This means that  $r \subset r_k = x_1 \dots x_L$ , i.e.  $r \in E_L$ .

Suppose that  $\widehat{\mathbb{G}}$  has property  $T$  and 2 is false. Let  $N = \text{Max}\{n_x \mid x \in E\}$  and  $\epsilon_n = \frac{1}{n^2 \sqrt{N^n}}$ . For all  $n \in \mathbb{N}^*$  there exists a unitary representation  $U_n$  of  $\widehat{\mathbb{G}}$  on a Hilbert space  $K_n$  with an  $(E, \epsilon_n)$ -invariant vector but without a non-zero invariant vector. Let  $\xi_n$  be a unit vector in  $K_n$  which is  $(E, \epsilon_n)$ -invariant. Write  $U_n = \sum_{y \in \text{Irred}(\widehat{\mathbb{G}})} U^{n,y}$  where  $U^{n,y}$  is a unitary element in  $\mathcal{B}(H_y) \otimes \mathcal{B}(K_n)$ . Let us show the following:

$$\|U^{n,y} \eta \otimes \xi_n - \eta \otimes \xi_n\|_{H_y \otimes K_n} < \frac{1}{n} \|\eta\|_{H_y}, \quad \forall n \in \mathbb{N}^*, \forall y \in E_n, \forall \eta \in H_y. \quad (3)$$

Let  $y \in E_n$  and  $t_y \in \text{Mor}(y, x_1 \dots x_n)$  such that  $t_y^* t_y = I_y$ . Note that, by the definition of a representation and using the description of the coproduct on  $\widehat{\mathbb{G}}$ , we have  $(t_y \otimes 1)U^{n,y} = U_{1,n+1}^{n,x_1} U_{2,n+1}^{n,x_2} \dots U_{n,n+1}^{n,x_n} (t_y \otimes 1)$  where the subscripts are used for the leg numbering notation. It follows that, for all  $\eta \in H_y$ , we have:

$$\begin{aligned} \|U^{n,y} \eta \otimes \xi_n - \eta \otimes \xi_n\| &= \|(t_y \otimes 1)U^{n,y} \eta \otimes \xi_n - (t_y \otimes 1)\eta \otimes \xi_n\| \\ &= \|U_{1,n+1}^{n,x_1} U_{2,n+1}^{n,x_2} \dots U_{n,n+1}^{n,x_n} t_y \eta \otimes \xi_n - t_y \eta \otimes \xi_n\| \\ &\leq \sum_{k=1}^n \|U_{k,n+1}^{n,x_k} t_y \eta \otimes \xi_n - t_y \eta \otimes \xi_n\|. \end{aligned}$$

Let  $(e_j^{x_i})_{1 \leq j \leq n_{x_i}}$  be an orthonormal basis of  $H_{x_i}$  and put

$$t_y \eta = \sum \lambda_{i_1 \dots i_n} e_{i_1}^{x_1} \otimes \dots \otimes e_{i_n}^{x_n}.$$

Then we have, for all  $y \in E_n$  and  $\eta \in H_y$ ,

$$\begin{aligned} \|U^{n,y} \eta \otimes \xi_n - \eta \otimes \xi_n\| &\leq \sum_k \left\| \sum_{i_1 \dots i_n} \lambda_{i_1 \dots i_n} (U_{k,n+1}^{n,x_k} e_{i_1}^{x_1} \otimes \dots \otimes e_{i_n}^{x_n} \otimes \xi_n - e_{i_1}^{x_1} \otimes \dots \otimes e_{i_n}^{x_n} \otimes \xi_n) \right\| \\ &\leq \sum_k \sum_{i_1 \dots i_n} |\lambda_{i_1 \dots i_n}| \|U_{k,n+1}^{n,x_k} e_{i_k}^{x_k} \otimes \xi_n - e_{i_k}^{x_k} \otimes \xi_n\| \\ &\leq n \epsilon_n \|t_y \eta\|_1, \end{aligned}$$

where  $\|t_y \eta\|_1 = \sum |\lambda_{i_1 \dots i_n}|$ . Note that  $\|t_y \eta\|_1 \leq \sqrt{N^n} \|\eta\|$ , thus we have

$$\begin{aligned} \|U^{n,y} \eta \otimes \xi_n - \eta \otimes \xi_n\| &\leq n \epsilon_n \sqrt{N^n} \|\eta\| \\ &\leq \frac{1}{n} \|\eta\|. \end{aligned}$$

This proves Eq. (3). It is now easy to finish the proof. Let  $U$  be the direct sum of the  $U_n$ . It is a unitary representation of  $\widehat{\mathbb{G}}$  on  $K = \bigoplus K_n$ . Let  $\delta > 0$  and  $L \subset \text{Irred}(\mathbb{G})$  a finite subset. Because  $\text{Irred}(\mathbb{G}) = \bigcup^\uparrow E_n$  there exists  $n_1$  such that  $L \subset E_n$  for all  $n \geq n_1$ . Choose  $n \geq n_1$  such that  $\frac{1}{n} < \delta$ . Put  $\xi = (0, \dots, 0, \xi_n, 0, \dots)$  where  $\xi_n$  appears in the  $n$ -th place. Let  $x \in L$  and  $\eta \in H_x$ . We have:

$$\begin{aligned} \|U^x \eta \otimes \xi - \eta \otimes \xi\| &= \|U^{n,x} \eta \otimes \xi_n - \eta \otimes \xi_n\| \\ &\leq \frac{1}{n} \|\eta\| < \delta \|\eta\|. \end{aligned}$$

Thus  $U$  has almost invariant vectors. It follows from property  $T$  that  $U$  has a non-zero invariant vector, say  $l = (l_n)$ . There is a  $n$  such that  $l_n \neq 0$  and from the  $U$ -invariance of  $l$  we conclude that  $l_n$  is  $U_n$ -invariant. This is a contradiction.  $\square$

Such a pair  $(E, \epsilon)$  as defined Proposition 9 is called a *Kazhdan pair* for  $\widehat{\mathbb{G}}$ . Let us give an obvious example of a Kazhdan pair.

**Proposition 10.** *Let  $\widehat{\mathbb{G}}$  be a finite-dimensional discrete quantum group. Then  $(\text{Irred}(\mathbb{G}), \sqrt{2})$  is a Kazhdan pair for  $\widehat{\mathbb{G}}$ .*

*Proof.* If  $\widehat{\mathbb{G}}$  is finite-dimensional then it is compact,  $\varphi$  is a trace and  $\hat{\varphi}$  is a normal functional. For  $x \in \text{Irred}(\mathbb{G})$  let  $(e_i^x)$  be an orthonormal basis of  $H_x$  and  $e_{ij}^x$  the associated matrix units. As  $Q_x = 1$ , we have  $\hat{\varphi}(e_{ij}^x) = \frac{\dim_q(x)^2}{n_x} \delta_{ij}$ . Let  $U \in l^\infty(\widehat{\mathbb{G}}) \otimes \mathcal{B}(K)$  be a unitary representation of  $\widehat{\mathbb{G}}$  with a unit vector  $\xi \in K$  such that:

$$\sup_{x \in \text{Irred}(\mathbb{G}), 1 \leq j \leq n_x} \|U^x e_j^x \otimes \xi - e_j^x \otimes \xi\| < \sqrt{2}.$$

Because  $\hat{\varphi}(1)^{-1}(\hat{\varphi} \otimes \text{id})(U)$  is the projection on the  $U$ -invariant vectors,  $\tilde{\xi} = (\hat{\varphi} \otimes \text{id})(U)\xi \in K$  is invariant. Let us show that  $\tilde{\xi}$  is non-zero. Writing  $U^x = \sum e_{ij}^x \otimes U_{ij}^x$  with  $U_{ij}^x \in \mathcal{B}(K)$ , we have:

$$\|U^x e_j^x \otimes \xi - e_j^x \otimes \xi\|^2 = 2 - 2\text{Re}\langle U_{jj}^x \xi, \xi \rangle, \quad \text{for all } x \in \text{Irred}(\mathbb{G}), 1 \leq j \leq n_x.$$

It follows that  $\text{Re}\langle U_{jj}^x \xi, \xi \rangle > 0$  for all  $x \in \text{Irred}(\mathbb{G})$  and all  $1 \leq j \leq n_x$ . Thus,

$$\text{Re}\langle \tilde{\xi}, \xi \rangle = \sum_{x,i,j} \text{Re}(\hat{\varphi}(e_{ij}^x) \langle U_{ij}^x \xi, \xi \rangle) = \sum_{x,i} \frac{\dim_q(x)^2}{n_x} \text{Re}(\langle U_{ii}^x \xi, \xi \rangle) > 0.$$

$\square$

**Remark 4.** It is easy to see that a discrete quantum group is amenable and has property  $T$  if and only if it is finite-dimensional. Indeed, the existence of almost invariant vectors for the regular representation is equivalent with amenability and it is well known that a discrete quantum group is finite dimensional if and only if the regular representation has a non-zero invariant vector. Moreover the previous proposition implies that all finite-dimensional discrete quantum groups have property  $T$ .

The main result of this paper is the following.

**Theorem 3.** *Let  $\widehat{\mathbb{G}}$  be discrete quantum group such that  $L^\infty(\mathbb{G})$  is an infinite dimensional factor. The following assertions are equivalent :*

1.  $\widehat{\mathbb{G}}$  has property  $T$ .
2.  $L^\infty(\mathbb{G})$  is a  $\text{II}_1$  factor with property  $T$ .

*Proof.* We can suppose that  $\mathbb{G}$  is reduced,  $C(\mathbb{G}) \subset \mathcal{B}(L^2(\mathbb{G}))$  and  $\mathbb{V} \in l^\infty(\widehat{\mathbb{G}}) \otimes L^\infty(\mathbb{G})$ . We denote by  $M$  the von Neumann algebra  $L^\infty(\mathbb{G})$ . For each  $x \in \text{Irred}(\mathbb{G})$  we choose an orthonormal basis  $(e_i^x)_{1 \leq i \leq n_x}$  of  $H_x$ . When  $\varphi$  is a trace we take  $e_i^{\bar{x}} = J_x(e_i^x)$ . We put  $u_{ij}^x = (\omega_{e_j^x, e_i^x} \otimes \text{id})(u^x)$ .

$1 \Rightarrow 2$  : Suppose that  $\widehat{\mathbb{G}}$  has property  $T$ . By Proposition 7,  $M$  is finite factor. Thus, it is a  $\text{II}_1$  factor. Let  $(E, \epsilon)$  be a Kazhdan pair for  $\widehat{\mathbb{G}}$ . Let  $K \in \mathcal{C}(M)$  with morphisms  $\pi_l : M \rightarrow \mathcal{B}(K)$  and  $\pi_r : M^{op} \rightarrow \mathcal{B}(K)$ . Let  $\delta = \frac{\epsilon}{\text{Max}\{n_x \sqrt{n_x}, x \in E\}}$ . Suppose that there exists a unit vector  $\xi' \in K$  such that:

$$\|u_{ij}^x \xi' - \xi' u_{ij}^x\| < \delta, \quad \forall x \in E, \forall 1 \leq i, j \leq n_x.$$

Define  $U = (\text{id} \otimes \pi_r)(\mathbb{V}^*)(\text{id} \otimes \pi_l)(\mathbb{V})$ . Because  $\mathbb{V}$  is a unitary representation of  $\widehat{\mathbb{G}}$  and  $\pi_r$  is an anti-homomorphism, it is easy to check that  $U$  is a unitary representation of  $\widehat{\mathbb{G}}$  on  $K$ . Moreover, for all  $x \in E$ , we have:

$$\begin{aligned} \|U^x e_i^x \otimes \xi' - e_i^x \otimes \xi'\| &= \|(\text{id} \otimes \pi_l)(u^x) e_i^x \otimes \xi' - (\text{id} \otimes \pi_r)(u^x) e_i^x \otimes \xi'\| \\ &= \left\| \sum_{k=1}^{n_x} e_k^x \otimes (u_{ki}^x \xi' - \xi' u_{ki}^x) \right\| \\ &\leq \sum_{k=1}^{n_x} \|e_k^x \otimes (u_{ki}^x \xi' - \xi' u_{ki}^x)\| \\ &< n_x \delta \leq \frac{\epsilon}{\sqrt{n_x}}. \end{aligned}$$

It follows easily that for all  $x \in E$  and all  $\eta \in H_x$  we have  $\|U^x \eta \otimes \xi' - \eta \otimes \xi'\| < \epsilon \|\eta\|$ . Thus there exists a non-zero  $U$ -invariant vector  $\xi \in K$ . It is easy to check that  $\xi$  is a central vector.

$2 \Rightarrow 1$  : Suppose that  $M$  is a  $\text{II}_1$  factor with property  $T$  and let  $\epsilon > 0$  and  $b_1, \dots, b_n \in M$  be as in Proposition 5. Let  $\varphi$  be the Haar state on  $\mathbb{G}$ . By [7], Theorem 8,  $\varphi$  is the unique tracial state on  $M$ . We can suppose that  $\|b_i\|_2 = 1$ . Using the classical G.N.S. construction  $(L^2(\mathbb{G}), \Omega)$  for  $\varphi$  we have, for all  $a \in M$ ,

$$a\Omega = \sum_{x,k,l} n_x \varphi((u_{kl}^x)^* a) u_{kl}^x \Omega.$$

In particular,  $\|b_i\|_2^2 = \sum n_x |\varphi((u_{kl}^x)^* b_i)|^2 = 1$ . Fix  $\delta > 0$  then there exists a finite subset  $E \subset \text{Irred}(\mathbb{G})$  such that, for all  $1 \leq i \leq n$ ,

$$\sum_{x \notin E, k, l} n_x |\varphi((u_{kl}^x)^* b_i)|^2 < \delta^2.$$

Let  $U$  be a unitary representation of  $\widehat{\mathbb{G}}$  on  $K$  having almost invariant vectors and  $\xi \in K$  an  $(E, \delta)$ -invariant unit vector. Turn  $L^2(\mathbb{G}) \otimes K$  into a correspondence from  $M$  to  $M$  using the morphisms  $\pi_l : M \rightarrow \mathcal{B}(L^2(\mathbb{G}) \otimes K)$ ,  $\pi_l(a) = U(a \otimes 1)U^*$  and  $\pi_r : M^{op} \rightarrow \mathcal{B}(L^2(\mathbb{G}) \otimes K)$ ,  $\pi_r(a) = Ja^*J \otimes 1$ , where  $J$  is the modular conjugation of  $\varphi$ . Let  $\widehat{\xi} = \Omega \otimes \xi$ . It is easy to see that  $\pi_l(u_{kl}^x) = \sum_s u_{ks}^x \otimes U_{sl}^x$  and, for all  $a \in M$ ,

$$a\widehat{\xi} = \sum n_x \varphi((u_{kl}^x)^* a) u_{ks}^x \Omega \otimes U_{sl}^x \xi.$$

Note that, because  $\varphi$  is a trace,  $\Omega$  is a central vector in  $L^2(\mathbb{G})$  and we have, for all  $a \in M$ ,  $\widehat{\xi}a = a\Omega \otimes \xi$ . It follows that, for all  $1 \leq i \leq n$ , we have

$$\begin{aligned} \|b_i \widehat{\xi} - \widehat{\xi} b_i\|^2 &= \left\| \sum_{x,k,l,s} n_x \varphi((u_{kl}^x)^* b_i) u_{ks}^x \Omega \otimes U_{sl}^x \xi - \sum_{x,k,l} n_x \varphi((u_{kl}^x)^* b_i) u_{kl}^x \Omega \otimes \xi \right\|^2 \\ &= \left\| \sum_{x,k,l} n_x \varphi((u_{kl}^x)^* b_i) \left( \sum_s u_{ks}^x \Omega \otimes U_{sl}^x \xi - u_{kl}^x \Omega \otimes \xi \right) \right\|^2 \\ &= \left\| \sum_{x,k,l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) \left( \sum_s e_s^x \otimes J_x(e_k^x) \otimes U_{sl}^x \xi - e_l^x \otimes J_x(e_k^x) \otimes \xi \right) \right\|^2 \\ &= \left\| \sum_{x,k,l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) J_x(e_k^x) \otimes \left( \sum_s e_s^x \otimes U_{sl}^x \xi - e_l^x \otimes \xi \right) \right\|^2 \\ &= \left\| \sum_{x,k,l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) J_x(e_k^x) \otimes (U^x e_l^x \otimes \xi - e_l^x \otimes \xi) \right\|^2 \\ &= \sum_{x,k} n_x \left\| \sum_l \varphi((u_{kl}^x)^* b_i) (U^x e_l^x \otimes \xi - e_l^x \otimes \xi) \right\|^2 \\ &= \sum_{x,k} n_x \|U^x \eta_k^x \otimes \xi - \eta_k^x \otimes \xi\|^2, \text{ where } \eta_k^x = \sum_l \varphi((u_{kl}^x)^* b_i) e_l^x \\ &= \sum_{x \in E, k} n_x \|U^x \eta_k^x \otimes \xi - \eta_k^x \otimes \xi\|^2 + \sum_{x \notin E, k} n_x \|U^x \eta_k^x \otimes \xi - \eta_k^x \otimes \xi\|^2 \\ &< \delta^2 \sum_{x \in E, k} n_x \|\eta_k^x\|^2 + 4 \sum_{x \notin E, k} n_x \|\eta_k^x\|^2 \\ &< \delta^2 \sum_{x \in E, k, l} n_x |\varphi((u_{kl}^x)^* b_i)|^2 + 4 \sum_{x \notin E, k, l} n_x |\varphi((u_{kl}^x)^* b_i)|^2 \\ &< \delta^2 + 4\delta^2 = 5\delta^2. \end{aligned}$$

By Proposition 5, for  $\delta$  small enough, there exists a central unit vector  $\widehat{\eta} \in L^2(\mathbb{G}) \otimes K$  with  $\|\widehat{\eta} - \widehat{\xi}\| < \sqrt{5}C\delta$ . Let  $P$  be the orthogonal projection on  $\mathbb{C}\Omega$ . If  $\delta$  is small enough then there is a non-zero  $\eta \in K$  such that  $(P \otimes 1)\widehat{\eta} = \Omega \otimes \eta$ . Write  $\widehat{\eta} = \sum_{y,s,t} e_t^y \otimes e_s^{\bar{y}} \otimes \eta_{s,t}^y$  where  $\eta_{s,t}^y \in K$  and  $\eta^1 = \eta$ . We have, for all



$x \in \text{Irred}(\mathbb{G})$  and all  $1 \leq i, j \leq n_x$ ,  $\pi_l(u_{ij}^x)\hat{\eta} = \pi_r(u_{ij}^x)\hat{\eta}$ . This means:

$$\sum_{k,y,t,s} u_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) \otimes U_{kj}^x \eta_{s,t}^y = \sum_{y,t,s} J(u_{ij}^x)^* J(e_t^y \otimes e_s^{\bar{y}}) \otimes \eta_{s,t}^y. \quad (4)$$

Let  $Q$  be the orthogonal projection on  $H_x \otimes H_{\bar{x}}$ . Using

$$u_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) \subset \bigoplus_{z \subset x \otimes y} H_z \otimes H_{\bar{z}},$$

and  $x \subset x \otimes y$  if and only if  $y = 1$ , we find:

$$Qu_{ik}^x(e_t^y \otimes e_s^{\bar{y}}) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e_k^x \otimes e_i^{\bar{x}}.$$

Using the same arguments and the fact that  $J = \bigoplus (J_x \otimes J_{\bar{x}})$  we find:

$$QJ(u_{ij}^x)^* J(e_t^y \otimes e_s^{\bar{y}}) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e_j^x \otimes e_i^{\bar{x}}.$$

Applying  $Q \otimes 1$  to Eq. (4) we obtain:

$$\sum_k e_k^x \otimes e_i^{\bar{x}} \otimes U_{kj}^x \eta = e_j^x \otimes e_i^{\bar{x}} \otimes \eta, \quad \text{for all } x \in \text{Irred}(\mathbb{G}), \ 1 \leq i, j \leq n_x.$$

Thus, for all  $x \in \text{Irred}(\mathbb{G})$  and all  $1 \leq j \leq n_x$ , we have:

$$U^x(e_j^x \otimes \eta) = \sum_k e_k^x \otimes U_{kj}^x \eta = e_j^x \otimes \eta.$$

Thus  $\eta$  is a non-zero  $U$ -invariant vector.  $\square$

The preceding theorem admits the following corollary about the persistence of property  $T$  by twisting.

**Corollary 2.** *Let  $\mathbb{G}$  be a compact quantum group such that  $L^\infty(\mathbb{G})$  is an infinite dimensional factor. Suppose that  $K$  is an abelian co-subgroup of  $\mathbb{G}$  (see [8]). Let  $\sigma$  be a continuous bicharacter on  $\hat{K}$  and denote by  $\mathbb{G}^\sigma$  the twisted quantum group. If  $\widehat{\mathbb{G}}$  has property  $T$  then  $\widehat{\mathbb{G}^\sigma}$  is a discrete quantum group with property  $T$ .*

*Proof.* If  $\widehat{\mathbb{G}}$  has property  $T$  then the Haar state  $\varphi$  on  $\mathbb{G}$  is a trace. Thus the co-subgroup  $K$  is stable (in the sense of [8]) and the Haar state  $\varphi_\sigma$  on  $\mathbb{G}^\sigma$  is the same, i.e.  $\varphi = \varphi_\sigma$ . It follows that  $\mathbb{G}^\sigma$  is a compact quantum group with  $L^\infty(\mathbb{G}^\sigma) = L^\infty(\mathbb{G})$ . Thus  $L^\infty(\mathbb{G}^\sigma)$  is a  $\text{II}_1$  factor with property  $T$  and  $\widehat{\mathbb{G}_\sigma}$  has property  $T$ .  $\square$

**Example 1.** The group  $SL_{2n+1}(\mathbb{Z})$  is I.C.C. and has property  $T$  for all  $n \geq 1$ . Let  $K_n$  be the subgroup of diagonal matrices in  $SL_{2n+1}(\mathbb{Z})$ . We have  $K_n = \mathbb{Z}_2^{2n} = \langle t_1, \dots, t_{2n} \mid t_i^2 = 1 \ \forall i, \ t_i t_j = t_j t_i \ \forall i, j \rangle$  and  $K_n$  is an abelian co-subgroup

of  $\mathbb{G}_{2n+1} = (C^*(SL_{2n+1}(\mathbb{Z})), \Delta)$ . Consider the following bicharacter on  $\widehat{K}_n = K_n$ :  $\sigma$  is the unique bicharacter such that  $\sigma(t_i, t_j) = -1$  if  $i \leq j$  and  $\sigma(t_i, t_j) = 1$  if  $i > j$ . By the preceding Corollary, the twisted quantum group  $\widehat{\mathbb{G}_{2n+1}^\sigma}$  has property  $T$  for all  $n \geq 1$ . When  $n$  is even,  $SL_n(\mathbb{Z})$  is not I.C.C. and  $I$  and  $-I$  lie in the centre of  $SL_n(\mathbb{Z})$ . We consider the group  $PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/\{I, -I\}$  in place of  $SL_n(\mathbb{Z})$  in the even case. It is well known that  $PSL_{2n}(\mathbb{Z})$  is I.C.C. and has property  $T$  for  $n \geq 2$ . The group of diagonal matrices in  $SL_{2n}(\mathbb{Z})$  is  $\mathbb{Z}_2^{2n-1}$  which contains  $\{I, -I\}$ . We consider the following abelian subgroup of  $PSL_{2n}(\mathbb{Z})$ :  $L_n = \mathbb{Z}_2^{2n-1}/\{I, -I\} = \mathbb{Z}_2^{2n-2} = K_{n-1}$  and the same bicharacter  $\sigma$  on  $K_{n-1}$ . Let  $\mathbb{G}_{2n} = (C^*(PSL_{2n}(\mathbb{Z})), \Delta)$ . By the preceding Corollary, the twisted quantum group  $\widehat{\mathbb{G}_{2n}^\sigma}$  has property  $T$  for all  $n \geq 2$ .

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